

II. Economics and Finance

- Economics: concerned with supply and demands, deals with how prices are determined.
- Finance: concerned with the borrowing and lending of money needed to engage in economic activity - here we will deal with the mathematical side of this.
- Thus: Finance is part of Economics

We differentiate between *price takers* (small economic agents) and *price makers* (big economic agents):

- *Price takers*: no power to influence prices but can enter the market without moving it against them. In the context of the stock market, individual investors are price-takers.
- *Price makers*: power to influence.

1. Trading

- Common items have prices at any given time, on the other hand items that are never bought do not have price (e.g. Buckingham Palace).
- Between the two, prices are known but not accurately: bid-ask spread (gap between the price at which a market participant will buy and at which he will sell).

a. Large trades:

- Prices jump as large trades affects the balance between supply and demand.
- The price is the level at which market clears (balance between supply and demand).
- Model such a market with the use of stochastic processes with jumps (Poisson process).

b. Small trades:

- From the distance:
Prices seem to move continuously - can be modeled with a stochastic process that is continuous (Brownian motion).
- Close up:
Lots of little jumps (all the individual small trades affecting the current balance between supply and demand). Such movement of prices is called *jitter*.

2. Utility

- Utility function measure how much utility - genuine use - money is to the economic agent.
- Income x give utility $U(x)$.
- Law of diminishing returns: U is strictly increasing but its graph is below the line $y=x$.
- Example: insurance contract (p.6).
- Guiding principle: each economic agent should try to maximize his expected utility (John Von Neumann & Oscar Morgenstern - 1947).

3. Loss

- Loss function can be thought as a negative of utility.
- Hence: one seeks to minimize one's expected loss.

4. Arbitrage

- Arbitrage opportunity: possibility to extract riskless profit from the market.
- Economic agent go to the market for various reasons: to insure - *hedge* - itself against adverse price movements that could affect their core business.
- *Speculators*, on the other hand, only interested in making a profit from a transaction.

- Should not be possible to extract money from the market without taking risk (all business activity is risky).
- Hence we make the assumption: **the market has No Arbitrage (NA)** - if they do exist, people will rush to exploit them and hence will dissipate them.
- Example of arbitrage free pricing:

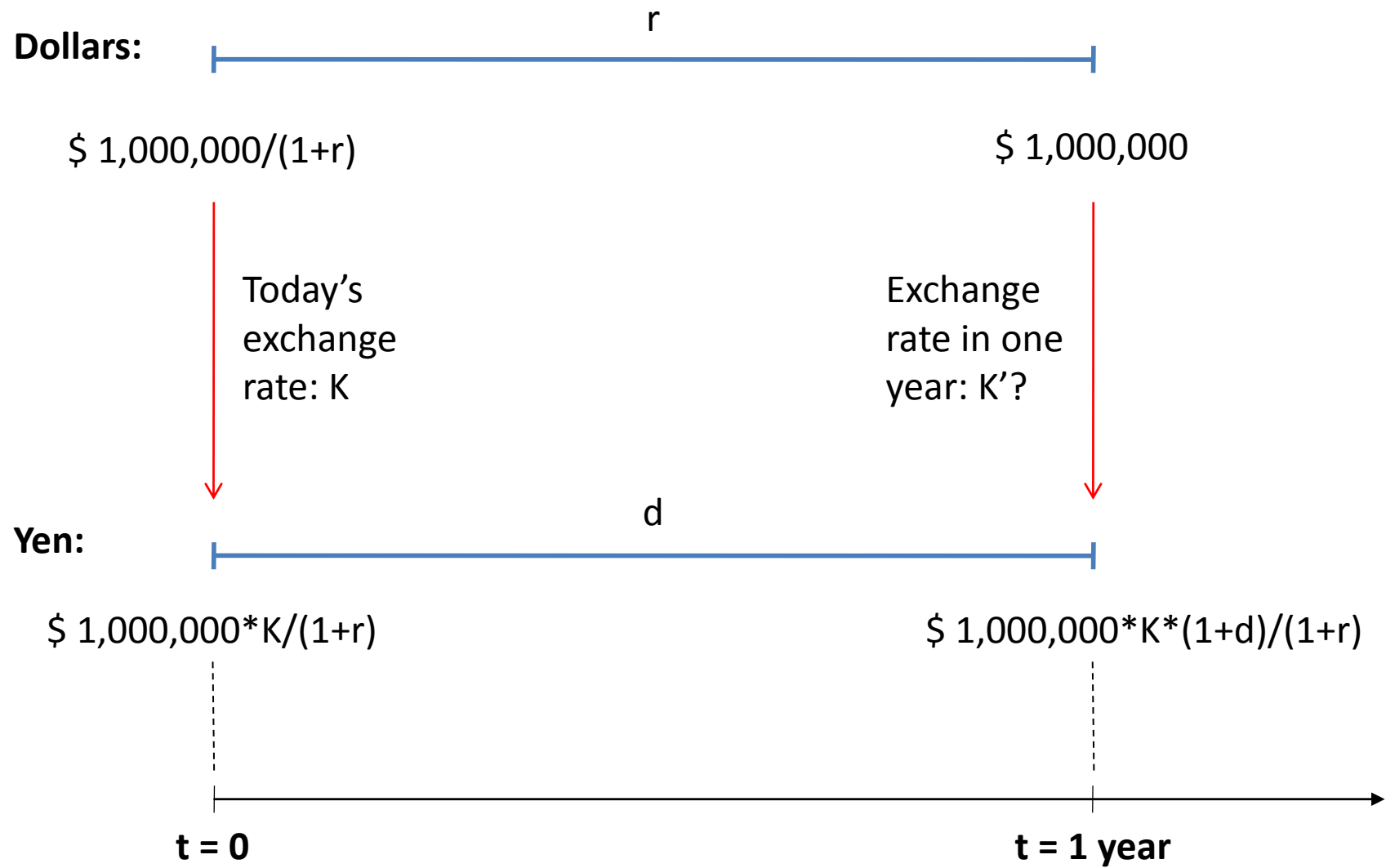
Suppose a company wishes to enter into a contract with you to exchange one million dollars for yen, one year from now. This is called a forward contract. Note that the company has no choice once it has entered the contract – this is not an option. How do we set the exchange rate?

We can do this by selling a zero coupon riskless bond in dollars today, which matures in a year with value one million dollars which corresponds precisely to the payment we get in a year.

The money we get in exchange for the bond today is one million divided by $1 + r$ where r is the yield of the bond.

We exchange that money at today's exchange rate and use the proceeds to buy a zero coupon bond maturing in a year in yen, which will of course grow according to the yield, d .

The value of this bond upon maturity will therefore be one million divided by $1 + r$, multiplied by the exchange rate K and then multiplied by $1 + d$ and it is precisely this amount which we give the company in exchange for one million dollars in a year from now.



- In conclusion, if today's exchange rate is K , then the forward exchange rate for a year in advance is:

$$K' = K^*(1+d)/(1+r)$$

The important point: any other rate leads to an arbitrage opportunity. In particular, if there is an alternative rate L available in the market we can make unlimited amounts of money by entering forward contracts at the rate L , and then hedging them using bonds as we have indicated above. Of course, whether we enter forward contracts buying or selling yen will depend on whether L is bigger or less than K .

In conclusion: L must be equal to K not because we believe the exchange rate in a year will be L , but because any other value leads to an arbitrage.

Note:

As well as illustrating arbitrage, this example really contains an important concept: *replication*.

If we can decompose an instrument into other instruments (with known prices) then we can price it simply by stringing those instruments together, any other price will lead to an arbitrage.

5. Idealized markets

In order to make the use of mathematics more bearable, various assumptions are commonly made. All are useful but valid to a first approximation only!

1. No Arbitrage (NA).
2. No transaction costs or taxes.
3. Same interest rate for borrowing and lending.
4. Unlimited liquidity (ability to turn good into money and vice versa).
5. No limitation of scale (unlimited quantity).

Market satisfying such assumptions will be called *perfect* (unrealistic but useful as a first approximation).

6. Mathematical finance: Phase 1 – Markowitz

- Finance was more an art than a science.
- 1952: thesis of Harry Markowitz introduces *portfolio theory*

Two key insights:

1. *Think of risk and return together*

No point investing in the stock market (risky) when one can invest risklessly (money in bank) - unless one expects the rate of return on the stock to be higher.

2. *Diversify, by holding a balanced portfolio with lots of negative correlation*

Investor is free to choose which sector of the economy to invest in. Prices may move against him. Insure against this by holding a balanced portfolio of assets from different sectors - chosen so that they move against each other (*negative correlation*)

Markowitz theory developed in the 1960s in **Capital Asset Pricing Model** (CAPM) - we will do this at the end of the week + refer to EPM945 Optimization and Decision Making.

7. Mathematical finance: Phase 2 - Black, Scholes and Merton

Definition - Option

Financial instrument giving one the right but not the obligation to make a specified transaction at a specified date at a specified time.

Question: *What is an option worth?*

- Before 1972: no answer to this question, the answer would depend on the agent's attitude to risk - i.e. his utility function.
- 1972: *Black-Scholes formula*, subject to the assumptions of an idealized market, one can price options!
Result was derived by showing that the option price satisfied a partial differential equation (PDE).

In fact: Options are redundant financial assets, they are equivalent to an appropriate combination of cash and stock (they are based on).

Hence knowing how much cash, how much stock and the current stock price, one can calculate the current option's price.

- 1981: J. M. Harrison & S. R. Pliska showed that the right mathematical machinery to use was *Martingales* with a particular type of calculus *Itô calculus*.